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Classical r-matrices and Poisson bracket structures on infinite-dimensional groups

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Starting with a canonical symplectic structure defined on the cotangent bundle T*G we derive, via Dirac hamiltonian reduction, Poisson brackets (PBs) on an arbitrary infinite-dimensional group G (admitting central extension). The PB structures are given in terms of an r-operator kernel related to the two-cocycle of the underlying Lie algebra and satisfying a differential classical Yang–Baxter equation. The explicit expressions of the PBs among the group variables for the $(N, 0)$ for $N=0, 1, ..., 4$ (super-) Virasoro groups and the group of area-preserving diffeomorphisms on the torus are presented.

1. Introduction

In the years following the discovery of the quantum group structures in the context of the quantum inverse scattering method [1] (for a recent review, see ref. [2]), their significance has been constantly increasing in various areas of theoretical physics ranging from completely integrable models [3–5] to conformal field theory [6,7] and quantum group gauge fields [8]. Already in the original papers [9] it was realized, that quantum groups are intimately related to Poisson bracket (PB) structures on ordinary Lie groups in a manner reminiscent of the relation between classical hamiltonian systems on symplectic manifolds and their quantum analogues.

Recently there have been a lot of discussions of PB structures on finite-dimensional semisimple Lie groups and their Kac–Moody generalizations, as well as their subsequent quantization [10]. In particular, the quantized group elements were identified with the chiral vertex operators in $D=2$ conformal models.

In view of the fundamental role played by PBs on ordinary groups, we propose in section 2 of the present letter a general geometric formalism for deriving PB structures on arbitrary infinite-dimensional groups with central extension. This formalism is based on the Dirac approach to the constrained hamiltonian systems. As a particular application of our method we obtain in sections 3 and 4 the fundamental PB relations between the group variables of the Virasoro group, N-extended super-Virasoro groups as well as $\widehat{\mathrm{SDiff}}\,(\mathrm{T}^2)$ – the group of area-preserving diffeomorphisms on the torus. The latter are the underlying symmetry groups of $(N, 0)$ $D=2$ induced (super-)gravity [11] and the toroidal membrane in the light-cone gauge [12], respectively.

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2. General formalism

It is a basic result of classical differential geometry [13] that the cotangent bundle $T^*\mathcal{M}$ of any riemannian manifold $\mathcal{M}$ possesses a symplectic structure on itself, i.e., any $T^*\mathcal{M}$ can be interpreted as a phase space of a classical hamiltonian mechanics with $\mathcal{M}$ being the configuration space. Taking $\mathcal{M}=\mathrm{G}$, where G is a Lie group, and denoting a generic point of $T^*\mathrm{G}$ as (U, g), one has the following canonical symplectic two-form:

$$\Omega(U,g) = -\mathrm{d}(\langle U|\,Y(g)\rangle) + \tfrac{1}{2}c\lambda\langle \hat{s}(Y(g))\,|\,Y(g)\rangle \tag{1}$$

(this is a generalization to centrally extended groups of the symplectic form used in ref. [7]; cf. ref. [14]).

In eq. (1) and below, the following notations are used. The canonical momentum U belongs to the cotangent space $T^*_g\mathrm{G}$ at the point $g\in\mathrm{G}$. $T^*_g\mathrm{G}$ can be identified with the dual space $\mathcal{G}^*$ of the Lie algebra $\mathcal{G}$ of G. The "pairing" between $\mathcal{G}$ and $\mathcal{G}^*$ is given by the bilinear form $\langle U|\xi\rangle$ for any $U\in\mathcal{G}^*$, $\xi\in\mathcal{G}$. $Y(g)$ in (1) is the fundamental $\mathcal{G}$-valued Maurer–Cartan one-form on G satisfying: $\mathrm{d}Y(g) = \frac{1}{2}[Y(g), Y(g)]$. In the last term in (1) we introduced the linear operator $\hat{s}$: $\mathcal{G}\to\mathcal{G}^*$ which defines a nontrivial two-cocycle on the Lie algebra $\mathcal{G}$:

$$\omega(\xi,\eta) \equiv -\lambda\langle\hat{s}(\xi)\,|\,\eta\rangle\,,\quad \forall\,\xi,\eta\in\mathcal{G}\,, \tag{2}$$

where λ is a numerical normalization constant. Eq. (2) yields a nontrivial central extension $\tilde{\mathcal{G}} = \mathcal{G}\oplus\mathbb{R}$ of $\mathcal{G}$ and, correspondingly, a central extension $\tilde{\mathcal{G}}^* = \mathcal{G}^*\oplus\mathbb{R}$ of the dual space $\mathcal{G}^*$. The parameter c in eq. (1) represents "central charge" of $\tilde{\mathcal{G}}^*$.

In what follows, we shall also need the explicit form of the adjoint and the coadjoint actions of G and $\tilde{\mathcal{G}}$ on the elements $(\xi, n)\in\tilde{\mathcal{G}}$ and $(U, c)\in\tilde{\mathcal{G}}^*$, respectively [14]:

$$\widetilde{\mathrm{Ad}}(g)(\xi,n) = (\mathrm{Ad}(g)\xi,\, n+\lambda\langle S(g^{-1})\,|\,\xi\rangle)\,, \tag{3}$$

$$\widetilde{\mathrm{ad}}(\xi_1,n_1)(\xi_2,n_2) \equiv [(\xi_1,n_1),(\xi_2,n_2)] = (\mathrm{ad}(\xi_1)\xi_2,\, -\lambda\langle\hat{s}(\xi_1)\,|\,\xi_2\rangle)\,, \tag{4}$$

$$\widetilde{\mathrm{Ad}}^*(g)(U,c) = (\mathrm{Ad}^*(g)U + c\lambda S(g),\, c)\,, \tag{5}$$

$$\widetilde{\mathrm{ad}}^*(\xi,n)(U,c) = (\mathrm{ad}^*(\xi)U + c\lambda\hat{s}(\xi),\, 0)\,. \tag{6}$$

Here $\mathrm{Ad}(g)$ and $\mathrm{ad}(\xi)$ [$\mathrm{Ad}^*(g)$ and $\mathrm{ad}^*(\xi)$] denote the ordinary (co)adjoint actions of G and $\mathcal{G}$ without the central extension [#1].

In eqs. (3) and (5) there appears another basic object – a nontrivial $\mathcal{G}^*$-valued one-cocycle $S(g)$ on G, which satisfies the relations [#2]

$$S(g_1g_2) = S(g_1) + \mathrm{Ad}^*(g_1)S(g_2)\,, \tag{7}$$

$$\hat{s}(\xi) = \frac{\mathrm{d}}{\mathrm{d}t}S(\mathrm{e}^{t\xi})\bigg|_{t=0}\,. \tag{8}$$

Let us note the following important relation between the group one-cocycle $S(g)$ and the fundamental Maurer–Cartan one-form $Y(g)$:

$$\mathrm{d}S(g) = \mathrm{ad}^*(Y(g))S(g) + \hat{s}(Y(g))\,, \tag{9}$$

and also the fundamental formula of Kirillov [15] expressing the group cocycle $S(g)$ in terms of the Lie-algebra cocycle operator $\hat{s}$:

[#1] They read explicitly: $\mathrm{Ad}(g)(\xi) = g\xi g^{-1}$, $\mathrm{ad}(\xi_1)\xi_2 = [\xi_1,\xi_2]$ and $\langle\mathrm{Ad}^*(g)U|\xi\rangle = \langle U|\mathrm{Ad}(g^{-1})\xi\rangle$, $\langle\mathrm{ad}^*(\xi)U|\eta\rangle = -\langle U|\mathrm{ad}(\xi)\eta\rangle$.

[#2] The physical interpretation of the $\mathcal{G}$-cocycle $\hat{s}$ is that of "anomaly" of the Lie algebra [i.e., existence of a c-number term in the commutator (4)], whereas the group cocycle $S(g)$ is the integrated "anomaly", i.e., the "anomaly" for finite group transformations [see eqs. (3) and (8)]. In the particular case of KM groups, $S(g) = \partial_x g g^{-1}$ and $Y(g^{-1}) = -g^{-1}\,\mathrm{d}g$ are combinations of the left and right KM currents $J_+ = \partial_+ g g^{-1}$, $J_- = g^{-1}\partial_- g$.

$$\mathrm{ad}^*(\xi)S(g)+\hat{s}(\xi)=\mathrm{Ad}^*(g)\hat{s}(\mathrm{Ad}(g^{-1})\xi)\ , \quad \forall\xi\in\mathscr{G},\, g\in\mathrm{G}\ . \tag{10}$$

Remark. The group- and algebra-cocycles $S(g)$ and $\hat{s}(\xi)$ can be generalized to include trivial (coboundary) parts:

$$\Sigma(g)\equiv\Sigma(g;\,(U_0,c))=c\lambda S(g)+\mathrm{Ad}^*(g)U_0-U_0\ , \tag{11}$$

$$\hat{\sigma}(\xi)\equiv\hat{\sigma}(\xi;\,(U_0,c))=\mathrm{ad}^*(\xi)U_0+c\lambda\hat{s}(\xi)=\frac{\mathrm{d}}{\mathrm{d}t}\Sigma(\mathrm{e}^{t\xi})\Big|_{t=0}\ , \tag{12}$$

where U_0 is an arbitrary point in the dual space $\mathscr{G}^*$ [#3]. The generalized cocycles (11) and (12) satisfy the same relations as (7), (9) and (10).

It is now easy to write down explicitly the Poisson brackets (PBs) among the canonical momenta and coordinates (U,g) corresponding to the symplectic two-form (1):

$$\{\langle U|\xi\rangle,\langle U|\eta\rangle\}_{\mathrm{PB}}=\langle\mathrm{ad}^*(\xi)U+c\lambda\hat{s}(\xi)\,|\,\eta\rangle\ , \tag{13}$$

$$\{\langle U|\xi\rangle,\Phi(g)\}_{\mathrm{PB}}=L_\xi\Phi(g)\equiv\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\mathrm{e}^{t\xi}g)\Big|_{t=0}\ , \tag{14}$$

$$\{\Phi_1(g),\Phi_2(g)\}_{\mathrm{PB}}=0\ . \tag{15}$$

Here ξ, η are arbitrary elements of $\mathscr{G}$, and $\Phi(g)$ together with $\Phi_{1,2}(g)$ are arbitrary smooth functions on G. In eq. (14) L_ξ denotes the left Lie derivative along the vector field corresponding to $\xi\in\mathscr{G}$.

In some cases below it will be useful to introduce a specific basis $\{T^\alpha\}$ of the Lie algebra $\mathscr{G}$ and its dual basis $\{T^*_\alpha\}$ in $\mathscr{G}^*$ [#4] and use the component expansions

$$[T^\alpha,T^\beta]=f^{\alpha\beta}_{\gamma}T^\gamma\ ,\quad \xi=\xi_\alpha T^\alpha\ ,\quad U=U^\alpha T^*_\alpha\ ,\quad \langle\hat{X}(\xi)\,|\,\eta\rangle=\xi_\alpha\hat{X}^{\alpha\beta}\eta_\beta \tag{16}$$

for arbitrary elements ξ, $\eta\in\mathscr{G}$, $U\in\mathscr{G}^*$ and an operator $\hat{X}$: $\mathscr{G}\to\mathscr{G}^*$.

Let us now consider a reduction of the original phase space T*G by the set of the following Dirac constraints:

$$\Psi_\xi(U,g)\equiv\langle\mathrm{Ad}^*(g^{-1})[U-\Sigma(g)-U_0]\,|\,\xi\rangle=0\ , \tag{17}$$

where $\Sigma(g)$ is the general one-cocycle on G defined in (11). Due to the cocycle property of $\Sigma(g)$ [cf. eq. (7)] we have the PBs

$$\{\langle U|\xi\rangle,\langle\Sigma(g)\,|\,\eta\rangle\}_{\mathrm{PB}}=\langle\mathrm{ad}^*(\xi)\Sigma(g)+\hat{\sigma}(\xi)\,|\,\eta\rangle\ , \tag{18}$$

which yields the PB algebra of constraints:

$$\langle\Psi_\xi(U,g),\Psi_\eta(U,g)\}_{\mathrm{PB}}=\Psi_{[\xi,\eta]}(U,g)-\langle\hat{\sigma}(\xi)\,|\,\eta\rangle\ . \tag{19}$$

Recalling eqs. (5) and (11), one observes that the reduced phase space $\mathscr{O}_{(U_0,c)}$,

$$\mathscr{O}_{(U_0,c)}\equiv\{(U,g)\in\mathrm{T^*G};\,U=U_0+\Sigma(g)=\mathrm{Ad}^*(g)U_0+c\lambda S(g)\}\ , \tag{20}$$

defined by the set of Dirac constraints (17) is a coadjoint orbit [13] of (the central extension of) G passing through the point (U_0,c) on the dual space $\tilde{\mathscr{G}}^*$. Clearly $\mathscr{O}_{(U_0,c)}\simeq\mathrm{G}/\mathrm{G}_{\mathrm{stat}}$ where $\mathrm{G}_{\mathrm{stat}}$ is the stationary subgroup of the point (U_0,c) with respect to the coadjoint action (5)

$$\mathrm{G}_{\mathrm{stat}}=\{k\in\mathrm{G};\,\Sigma(k)=0\}\ . \tag{21}$$

[#3] Sometimes we shall suppress for brevity the dependence on (U_0,c) of $\Sigma(g)$ (11) and $\hat{\sigma}(\xi)$ (12).

[#4] $\langle T^*_\alpha\,|\,T^\beta\rangle=\delta^\beta_\alpha$ and the Lie-algebra indices α are of the form: $\alpha=(A;\,(x_1,\ldots,x_p))$, where A are discrete indices related to finite-dimensional Lie algebras (as in the case of Kac–Moody algebras) and $(x_1,\ldots,x_p)$ are continious parameters.

The Lie algebra corresponding to $G_{\rm stat}$ is

$$\mathscr{G}_{\rm stat} \equiv \{\xi_0 \in \mathscr{G};\ \hat{\sigma}(\xi_0)=0\}\ . \tag{22}$$

From (22) we deduce that the Dirac constraints $\Psi_\xi(U,g)$ (17) are combinations of second-class $\Psi_\perp \equiv \Psi_{\xi\perp}(U,g)$ and first-class $\Psi_0 \equiv \Psi_{\xi 0}(U,g)$, where $\xi=\xi_0+\xi_\perp$ with $\xi_0 \in \mathscr{G}_{\rm stat}$ and $\xi_\perp \in \mathscr{G} \setminus \mathscr{G}_{\rm stat}$.

Let us now compute the Dirac brackets (DBs) between smooth functions $\Phi_{1,2}(g)$, i.e., the PBs between $\Phi_{1,2}(g)$ on the reduced phase space $\mathscr{O}_{(U_0,c)}$ (20). Taking into account (13)–(15), (18), (19) and (22) we find

$$\{\Phi_1(g),\Phi_2(g)\}_{\rm DB}=\{\Phi_1(g),\Psi^\alpha(U,g)\}_{\rm PB}(\hat{\sigma}_\perp^{-1})_{\alpha\beta}\{\Psi^\beta(U,g),\Phi_2(g)\}_{\rm PB}=-r_{\alpha\beta}R^\alpha\Phi_1(g)R^\beta\Phi_2(g)\ . \tag{23}$$

Here, for simplicity, we have used the component notations (16). Further

$$R_\xi\Phi(g)=\xi_\alpha R^\alpha\Phi(g)=L_{{\rm Ad}(g)\xi}\Phi(g)=\frac{\mathrm{d}}{\mathrm{d}t}\Phi(g\,\mathrm{e}^{t\xi})\Big|_{t=0} \tag{24}$$

denotes the right Lie derivative along ξ and

$$r_{\alpha\beta}=(\hat{\sigma}_\perp^{-1})_{\alpha\beta} \tag{25}$$

is the operator kernel of the inverse operator of the cocycle operator $\hat{\sigma}$: $\mathscr{G}\to\mathscr{G}^*$ (12) restricted to the nonzero-mode subspace $\mathscr{G}\setminus\mathscr{G}_{\rm stat}$.

It is a simple exercise to show that the Jacobi identities for the Dirac brackets (23) imply the following equation obeyed by the kernel $r_{\alpha\beta}$ (25) [#5]:

$$r_{\alpha\delta}f^{\delta\gamma}_{\beta}r_{\gamma\rho}+r_{\rho\delta}f^{\delta\gamma}_{\alpha}r_{\gamma\beta}+r_{\beta\delta}f^{\delta\gamma}_{\rho}r_{\gamma\alpha}=0\ . \tag{26}$$

Let us introduce the matrix $r\equiv r_{\alpha\beta}T^\alpha\otimes T^\beta\in\mathscr{G}\otimes\mathscr{G}$ and denote by $r^{(12)}\equiv r_{\alpha\beta}T^\alpha\otimes T^\beta\otimes\mathbb{1}\in\mathscr{U}(\mathscr{G})\otimes\mathscr{U}(\mathscr{G})\otimes\mathscr{U}(\mathscr{G})$ the imbedding of $r\in\mathscr{G}\otimes\mathscr{G}$ into $\mathscr{U}(\mathscr{G})\otimes\mathscr{U}(\mathscr{G})\otimes\mathscr{U}(\mathscr{G})$ and similarly for $r^{(13)}$ and $r^{(23)}$, where $\mathscr{U}(\mathscr{G})$ denotes the universal envelopping algebra of $\mathscr{G}$. Then one can rewrite the Jacobi identities (26) in the form of the well-known classical Yang–Baxter equation (CYBE) [1]:

$$[r^{(12)},r^{(13)}]+[r^{(12)},r^{(23)}]+[r^{(13)},r^{(23)}]=0\ ,\quad \text{with } r^{(12)}=-r^{(21)}\ . \tag{27}$$

The identification (25) of the inverse of the r-matrix, satisfying CYBE (27), with the two-cocycle on the underlying Lie algebra appeared for the first time in ref. [4].

Note that the group cocycle $\Sigma(g)$ (11) generates left group translations on the orbit $\mathscr{O}_{(U_0,c)}$ (20) as it follows upon substituting $\Phi_{1,2}(g)=\langle\Sigma(g)|\xi_{1,2}\rangle$ into (23) (cf. ref. [14]).

One can impose another set of Dirac constraints:

$$\tilde{\Psi}_\xi(U,g)\equiv\langle U-U_0|\xi\rangle=0 \tag{28}$$

and consider a reduction of the original phase space T*G with respect to (28) instead of (17). Calculation of the new DBs gives

$$\{\Phi_1(g),\Phi_2(g)\}_{\widetilde{\rm DB}}=r_{\alpha\beta}L^\alpha\Phi_1(g)L^\beta\Phi_2(g)\ , \tag{29}$$

with left Lie derivatives on the RHS. Eqs. (23) and (29) yield the fundamental PBs of the corresponding geometric actions $W[g]$ and $\tilde{W}[g]$ on the reduced phase spaces $\mathscr{O}_{(U_0,c)}$ (20) and $\tilde{\mathscr{O}}_{(U_0,c)}$ defined by the Dirac constraints (17) and (28), respectively [#6]. It is easy to show that $\tilde{W}[g]=-W[g^{-1}]$, i.e., $\tilde{W}[g]$ is the Legendre transform of $W[g]$ [16] where

[#5] We use here the commutation relation $R_\xi R_\eta-R_\eta R_\xi=R_{[\xi,\eta]}$ for the right Lie derivative.

[#6] The action on the original "large" phase space T*G with symplectic structure (1) is $W[U,g]=\int \mathrm{d}^{-1}\Omega(U,g)$.

$$W[g] = -\int [\langle \Sigma(g) | Y(g) \rangle - \tfrac{1}{2} d^{-1} (\langle \hat{\sigma}(Y(g)) | Y(g) \rangle)] . \tag{30}$$

3. Applications: Poisson bracket structures for $(N, 0)$ $D=2$ (super-) gravities

Let us first apply the general formalism of section 2 to the case of Virasoro group. Its group elements $g \simeq F(x)$ are smooth diffeomorphisms of the circle S^1. Group multiplication is given by composition of diffeomorphisms in inverse order, e.g., $g_1 \cdot g_2 = F_2 \circ F_1(x) = F_2(F_1(x))$. The basic objects in (3)–(6) have now the following explicit form (see, e.g. ref. [17]):

$$\mathrm{Ad}(F)\xi = (\partial_x F)^{-1} \xi(F(x)) , \quad \mathrm{Ad}^*(F) U = (\partial_x F)^2 U(F(x)) ,$$

$$\mathrm{ad}(\xi)\eta \equiv [\xi, \eta] = \xi \partial_x \eta - (\partial_x \xi)\eta , \quad \mathrm{ad}^*(\xi) U = \xi \partial_x U + 2(\partial_x \xi) U , \tag{31}$$

$$\hat{s}(\xi) = \partial_x^3 \xi , \quad S(F) = \frac{\partial_x^3 F}{\partial_x F} - \frac{3}{2} \left(\frac{\partial_x^2 F}{\partial_x F} \right)^2 . \tag{32}$$

Here $S(F)$ is the well-known schwarzian.

Using (32) it is easy to calculate the right Lie derivative (24):

$$R_\xi \Phi[F] = \int \mathrm{d}x \; \xi(x) R_x \Phi[F] , \quad R_x \Phi[F] \equiv \frac{1}{\partial_y F} \frac{\delta \Phi[F]}{\delta F(y)} \bigg|_{y = F^{-1}(x)} . \tag{33}$$

The r-matrix $\|r_{\alpha\beta}\|$ is now represented by an operator kernel $r(x, y)$:

$$r = r_{\alpha\beta} T^\alpha \otimes T^\beta = \int \mathrm{d}x \, \mathrm{d}y \; r(x, y) \hat{\mathcal{T}}(x) \hat{\mathcal{T}}(y) , \tag{34}$$

where $\{\hat{\mathcal{T}}(x)\}$ indicate a basis in the Virasoro Lie algebra [#7].

In what follows, for simplicity, we will put to zero the generic point U_0 of the dual space $\mathscr{G}^*$ which parametrizes the reduced phase space (20) and the inverse r-operator (12). In such a case the r-operator satisfies the differential equations [cf. (26) and (25)]:

$$c\lambda \, \partial_x^3 r(x, y) = \delta(x-y) , \quad r(x, y) = -r(y, x) ,$$

$$\sum_{\mathrm{cyclic}(1,2,3)} [r(x_1, x_2) \, \partial_{x_2} r(x_2, x_3) - \partial_{x_2} r(x_1, x_2) r(x_2, x_3)] = 0 , \tag{35}$$

the latter one being the CYBE for the Virasoro group. The normalization constant $\lambda = -1/24\pi$ [this is true for all $(N, 0)$ (super-) Virasoro groups]. From (35) one finds

$$r(x, y) = \frac{1}{c\lambda} [\tfrac{1}{4}(x-y)^2 \varepsilon(x-y) + b_0(x^2 - y^2) + b_1(x-y) + b_2 xy(x-y)] , \tag{36}$$

where b_0, b_1, b_2 are arbitrary constants subject to the constraint $b_0^2 - b_1 b_2 = \frac{1}{16}$. It is easy to check that $r(x, y)$ preserves its form (36) under an $SL(2; \mathbb{R})$ fractional-linear transformation on x and y. Note that the constants b_1 and b_2 in (36) are dimensionful. In what follows, we shall choose them equal to zero.

The general DBs for smooth functions of the group elements (23) now specialize to

$$\{\Phi_1[F], \Phi_2[F]\}_{\mathrm{DB}} = -\int \mathrm{d}x \, \mathrm{d}y \frac{\delta \Phi_1}{\delta F(x)} r(F(x), F(y)) \frac{\delta \Phi_2}{\delta F(y)} . \tag{37}$$

[#7] With commutation relations $[\hat{\mathcal{T}}(x), \hat{\mathcal{T}}(y)] = 2\hat{\mathcal{T}}(x) \, \partial_x \delta(x-y) + \partial_x \hat{\mathcal{T}}(x) \delta(x-y) - (1/24\pi) \partial_x^3 \delta(x-y)$.

In particular, accounting for (36) we reproduce the result of refs. [5,2] for the fundamental brackets between the Virasoro group elements:

$$\{F(x), F(y)\}_{\mathrm{DB}} = -r(F(x), F(y)) = -\frac{1}{4c\lambda}\{[F(x)-F(y)]^2\varepsilon(x-y)+[F^2(x)-F^2(y)]\} . \tag{38}$$

Let us now generalize this result to the case of $(N, 0)$ extended super-Virasoro groups for any $N\leqslant 4$. First, let us recall the explicit form of the basic objects in a manifestly $(N, 0)$ supersymmetric formalism [18]. The points of the $(N, 0)$ superspace are labeled as (t, z), $z\equiv(x, \theta^i)$, $i=1, ..., N$. Taken in a specific basis $\{\hat{\mathscr{T}}(z)\}$ the $(N, 0)$ super-Virasoro Lie algebra has the form [#8]

$$[\hat{\mathscr{T}}(z_1), \hat{\mathscr{T}}(z_2)\} = (-1)^N(2-\tfrac{1}{2}N)\hat{\mathscr{T}}(z_1)\,\partial_{x_1}\delta^{(N)}(z_1-z_2) + (-1)^N\partial_{x_1}\hat{\mathscr{T}}(z_1)\delta^{(N)}(z_1-z_2)$$

$$+\tfrac{1}{2}\mathrm{i}\,\mathrm{D}_i\hat{\mathscr{T}}(z_1)\,\mathrm{D}^i\delta^{(N)}(z_1-z_2) - \frac{1}{24\pi}\mathrm{i}^{N(N-2)}\,\mathrm{D}^N\partial_x^{3-N}\delta^{(N)}(z_1-z_2) . \tag{39}$$

The group elements are given by superconformal diffeomorphisms:

$$z\equiv(x, \theta^j)\to\tilde{Z}\equiv(F(x, \theta^j), \tilde{\Theta}^i(x, \theta^j)) \tag{40}$$

obeying the following constraints:

$$\mathrm{D}^jF - \mathrm{i}\tilde{\Theta}^k\,\mathrm{D}^j\tilde{\Theta}_k = 0 , \quad \mathrm{D}^j\tilde{\Theta}^l\,\mathrm{D}^k\tilde{\Theta}_l - \delta^{jk}[\mathrm{D}\tilde{\Theta}]^2_N = 0 , \quad [\mathrm{D}\tilde{\Theta}]^2_N \equiv \frac{1}{N}\mathrm{D}^m\tilde{\Theta}^n\,\mathrm{D}_m\tilde{\Theta}_n . \tag{41}$$

These constraints are imposed by requiring covariance of the super-derivative D^j: $\mathrm{D}^j\to(\mathrm{D}^j\tilde{\Theta}_k)\,\tilde{\mathrm{D}}^k$ under (40). The $(N, 0)$ supersymmetric analogues of (32) read

$$\mathrm{Ad}(\tilde{Z})\xi = ([\mathrm{D}\tilde{\Theta}]^2_N)^{-1}\xi(\tilde{Z}(z)) , \quad \mathrm{Ad}^*(\tilde{Z})B = ([\mathrm{D}\tilde{\Theta}]^2_N)^{2-N/2}B(\tilde{Z}(z)) ,$$

$$\mathrm{ad}(\xi)\eta\equiv[\xi, \eta] = \xi\,\partial_x\eta - (\partial_x\xi)\eta - \tfrac{1}{2}\mathrm{i}\,\mathrm{D}_k\xi\,\mathrm{D}^k\eta, \quad \mathrm{ad}^*(\xi)B = \xi\,\partial_xB + (2-\tfrac{1}{2}N)(\partial_x\xi)B - \tfrac{1}{2}\mathrm{i}\,\mathrm{D}_k\xi\,\mathrm{D}^kB ,$$

$$\hat{s}(\xi) = \mathrm{i}^{N(N-2)}\mathrm{D}^N\partial_x^{3-N}\xi . \tag{42}$$

The associated $\mathscr{G}^*$-valued group one-cocycles $S_N(\tilde{Z})$ coincide with the well-known [18] $(N, 0)$ super-schwarzians.

Let us now derive the explicit form of the Dirac brackets (23) for the $(N, 0)$ super-Virasoro group. First, we find the action of the right Lie derivative:

$$R_\xi[\tilde{Z}] = \int(\mathrm{d}z)\xi(z)R_z\Phi[\tilde{Z}] ,$$

$$R_z\Phi[\tilde{Z}] \equiv \left(\frac{1}{([\mathrm{D}'\tilde{\Theta}]^2_N)^2}\,[(1-\tfrac{1}{2}N)\,\partial_{x'}\tilde{Z} - (-1)^{\varepsilon(\tilde{Z})}\cdot\tfrac{1}{2}\mathrm{i}\,\mathrm{D}'_k\tilde{Z}\,\mathrm{D}'^k]\,\frac{\delta\Phi}{\delta\tilde{Z}(z')}\right)\Bigg|_{z'\equiv\tilde{Z}^{-1}(z)} . \tag{43}$$

The $(N, 0)$ supersymmetric r-operator kernel satisfies [cf. (25), (42)]

$$\mathrm{i}^{N(N-2)}c\lambda\,\mathrm{D}_1^N\partial_{x_1}^{3-N}r_N(z_1, z_2) = \delta^{(N)}(z_1-z_2)\equiv\delta(x_1-x_2)\delta^{(N)}(\theta_1-\theta_2) \tag{44}$$

and obeys the $(N, 0)$ supersymmetric CYBE:

$$\sum_{\mathrm{cyclic}(1,2,3)}[r_N(z_1, z_2)\,\partial_{x_2}r_N(z_2, z_3) - \partial_{x_2}r_N(z_1, z_2)r_N(z_2, z_3) - \tfrac{1}{2}\mathrm{i}\,\mathrm{D}_2^jr_N(z_1, z_2)\,\mathrm{D}_{2j}r_N(z_2, z_3)] = 0 . \tag{45}$$

The general solution of (44), accounting for (45), reads (as in the $N=0$ case, we discard the terms with dimensionful constants)

[#8] The following superspace notations are used: $\mathrm{D}^i = \partial/\partial\theta_i + \mathrm{i}\theta^i\,\partial_x$, $\mathrm{D}^N\equiv(1/N!)\epsilon_{i_1\ldots i_N}\mathrm{D}^{i_1}\ldots\mathrm{D}^{i_N}$.

$$r_N(z_1, z_2) = \frac{1}{4c\lambda}\,[\varepsilon(x_1 - x_2 - i\theta_1^k\theta_{2k}) + x_1^2 - x_2^2 - 2i\theta_1^k\theta_{2k}(x_1 + x_2)]\,. \tag{46}$$

Now, inserting the explicit expressions (43) and (46) into the general DB formula (23), one obtains the following fundamental brackets for the $(N, 0)$ super-Virasoro group parameters:

$$\{F(z_1), F(z_2)\}_{\mathrm{DB}} = -(1 - \tfrac{1}{2}\tilde{\Theta}_1^l\tilde{\mathrm{D}}_{1l})(1 - \tfrac{1}{2}\tilde{\Theta}_2^k\tilde{\mathrm{D}}_{2k})r_N(\tilde{Z}(z_1), \tilde{Z}(z_2))\,, \tag{47}$$

$$\{F(z_1), \tilde{\Theta}^k(z_2)\}_{\mathrm{DB}} = \tfrac{1}{2}i(1 - \tfrac{1}{2}\tilde{\Theta}_1^l\tilde{\mathrm{D}}_{1l})\tilde{\mathrm{D}}_{2k}r_N(\tilde{Z}(z_1), \tilde{Z}(z_2))\,, \tag{48}$$

$$\{\tilde{\Theta}^k(z_1), \tilde{\Theta}^l(z_2)\}_{\mathrm{DB}} = \tfrac{1}{4}\tilde{\mathrm{D}}_1^k\tilde{\mathrm{D}}_2^l r_N(\tilde{Z}(z_1), \tilde{Z}(z_2))\,, \tag{49}$$

where the $(N, 0)$ supersymmetric r-operator kernel $r_N(\ ,\ )$ is given by (46) and the following notation was used:

$$\tilde{\mathrm{D}}_{1,2}^k \equiv \left(\frac{\partial}{\partial\tilde{\Theta}^k} + i\tilde{\Theta}^k\,\partial_{\tilde{X}}\right)\Bigg|_{\tilde{\Theta}^k \equiv \tilde{\Theta}^k(z_{1,2}),\ \tilde{X} \equiv F(z_{1,2})}\,. \tag{50}$$

We note that (47)–(49) are compatible with the superconformal constraints (41). In the particular cases $N=1,2$, the PB-algebra (47)–(49) was derived in ref. [19].

As a consequence of the fundamental brackets (49) we get the following simple free-field brackets for the unconstrained $(N, 0)$ superfield $\phi(z)$:

$$\phi(z) \equiv \ln([\mathrm{D}\tilde{\Theta}]_N^2(z))\,,$$

$$\{\phi(z_1), \phi(z_2)\}_{\mathrm{DB}} = \frac{1}{2c\lambda}\,\varepsilon(x_1 - x_2 - i\theta_1^k\theta_{2k})\,. \tag{51}$$

From (51) we easily find the geometric action on the reduced phase space $\mathscr{C}_{(U_0,c)}$ – the coadjoint orbit of $(N, 0)$ super-Virasoro, which produces the fundamental brackets (47)–(49)

$$W_N[\tilde{Z}] = i^{N(N-2)}c\lambda \int \mathrm{d}t\ (\mathrm{d}z)\ \partial_t(\ln[\mathrm{D}\tilde{\Theta}]_N^2)\ \mathrm{D}^N\partial_x^{1-N}([\mathrm{D}\tilde{\Theta}]_N^2)\,. \tag{52}$$

For $N=1, 2$ (52) reduces to the known Polyakov geometric actions for $N=1, 2$ induced $D=2$ supergravity [20,14] which are *local* in ordinary space. For $N=3, 4$, however, the super-Virasoro geometric action (52) is nonlocal with respect to x. In order to have both local and unconstrained off-shell $(N, 0)$ $D=2$ superspace formulation when $N \geqslant 3$, one needs superfields defined on extended harmonic superspace [21].

4. Application: Poisson brackets for the Wess–Zumino action of toroidal membrane

In ref. [16] it was shown that the Wess–Zumino (WZ) anomalous effective action for the toroidal membrane in the light-cone gauge is precisely the geometric action on a generic coadjoint orbit of $\widehat{\mathrm{SDiff}}\,(\mathrm{T}^2)$ [22]. The latter is the centrally-extended group of area-preserving diffeomorphisms on the torus T^2 [12]. The elements of $\widehat{\mathrm{SDiff}}\,(\mathrm{T}^2)$ are described by smooth diffeomorphisms $\mathrm{T}^2 \ni \boldsymbol{x} \equiv (x^1, x^2) \to F^i(\boldsymbol{x}) \in \mathrm{T}^2$ $(i=1, 2)$, such that

$$\det\left\|\frac{\partial F^i}{\partial x^j}\right\| = 1 \quad \text{or} \quad \epsilon^{kl}\,\partial_k F^i\,\partial_l F^j = \epsilon^{ij}\,. \tag{53}$$

The Lie algebra of $\widehat{\mathrm{SDiff}}\,(\mathrm{T}^2)$ reads

$$[\hat{\mathscr{L}}(\boldsymbol{x}), \hat{\mathscr{L}}(\boldsymbol{y})] = -\epsilon^{ij}\,\partial_i\hat{\mathscr{L}}(\boldsymbol{x})\,\partial_j\delta^{(2)}(\boldsymbol{x}-\boldsymbol{y}) - a^i\partial_i\delta^{(2)}(\boldsymbol{x}-\boldsymbol{y})\,, \tag{54}$$

where $\boldsymbol{a} \equiv (a^1, a^2)$ are the "central charges" [12].

Now, the r-operator kernel for $\widetilde{\mathrm{SDiff}}\,(\mathrm{T}^2)$ obeys the equations [cf. (35)]

$$a_i\, \partial_x^i r(\boldsymbol{x}, \boldsymbol{y}) = \delta^{(2)}(\boldsymbol{x}-\boldsymbol{y}) , \qquad \sum_{\mathrm{cyclic}(1,2,3)} \epsilon_{kl}\, \partial_{x2}^k r(\boldsymbol{x}_1, \boldsymbol{x}_2)\, \partial_{x2}^l r(\boldsymbol{x}_2, \boldsymbol{x}_3) = 0 , \tag{55}$$

where the second equation in (55) is the CYBE for $\widetilde{\mathrm{SDiff}}\,(\mathrm{T}^2)$. The solution to (55) reads

$$r(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{2\sqrt{\boldsymbol{a}^2}}\, \varepsilon(x_\parallel - y_\parallel)\delta(x_\perp - y_\perp) , \tag{56}$$

where we introduced the longitudinal and transverse projections of a two-dimensional vector $\boldsymbol{C}$ along $\boldsymbol{a}$:

$$C_\parallel = \frac{a^i C_i}{\sqrt{\boldsymbol{a}^2}} , \quad C_\perp = \frac{a^i \epsilon_{ij} C^j}{\sqrt{\boldsymbol{a}^2}} . \tag{57}$$

Plugging (56) into (23) we get

$$\{F^i(\boldsymbol{x}_1), F^j(\boldsymbol{x}_2)\}_{\mathrm{DB}} = \epsilon^{ik}\epsilon^{jl} \frac{\partial}{\partial F_1^k} \frac{\partial}{\partial F_2^l} r(\boldsymbol{F}_1, \boldsymbol{F}_2)\bigg|_{\boldsymbol{F}_{1,2} = \boldsymbol{F}(\boldsymbol{x}_{1,2})} . \tag{58}$$

In particular, for the "longitudinal" group variables $F_\parallel(\boldsymbol{x}) = (a^i/\sqrt{\boldsymbol{a}^2}) F_i(\boldsymbol{x})$ we obtain the quadratic PBs for $\widetilde{\mathrm{SDiff}}\,(\mathrm{T}^2)$:

$$\{F_\parallel(\boldsymbol{x}), F_\parallel(\boldsymbol{y})\}_{\mathrm{DB}} = -\frac{1}{2\sqrt{\boldsymbol{a}^2}}\, \varepsilon(x_\parallel - y_\parallel) \frac{\partial F_\parallel(\boldsymbol{x})}{\partial x_\parallel} \frac{\partial F_\parallel(\boldsymbol{y})}{\partial y_\parallel} \frac{\partial^2}{\partial x_\perp^2} \delta(x_\perp - y_\perp) . \tag{59}$$

Eqs. (58), (59) are the PBs corresponding to the WZ membrane action [22,16]:

$$W[\boldsymbol{F}] = -\tfrac{1}{3} \int \mathrm{d}t\, \mathrm{d}^2x (a^k \epsilon_{kl} F^l) \epsilon_{ij} F^i\, \partial_t F^j , \tag{60}$$

where $\boldsymbol{F}$ are constrained by (53).

5. Outlook and discussion

It was shown in ref. [9] that the classical "limit" of a quantum group is a Lie–Poisson group, on which the PB structure is given by a sum of PBs (23) and (29):

$$\{\Phi_1(h), \Phi_2(h)\}_{\mathrm{LP}} = r_{\alpha\beta}(L^\alpha \Phi_1(h) L^\beta \Phi_2(h) - R^\alpha \Phi_1(h) R^\beta \Phi_2(h)) , \quad h \in \mathrm{G} . \tag{61}$$

Recently, a new approach has been proposed in ref. [7] for constructing Lie–Poisson structures (61) and their subsequent quantization starting with the phase space T*G. This formalism uses a special change of variables which, when generalized to arbitrary infinite-dimensional groups, reads (in notations of section 2) as $(U, g) \to (g_+, g_-, U_0, k_0)$:

$$g = g_+ k_0 g_- , \quad U = \Sigma(g_+; (U_0, c)) + U_0 . \tag{62}$$

Here g_+ belongs to the right coset space $\mathrm{G}/\mathrm{G}_{\mathrm{stat}} \simeq \mathcal{O}_{(U_0,c)}$ [i.e., the coadjoint orbit (20)], g_- belongs to the left coset $\mathrm{G}_{\mathrm{stat}} \backslash \mathrm{G}$, $k_0 \in \mathrm{G}_{\mathrm{stat}}$ while $U_0 \in \mathscr{G}^*_{\mathrm{stat}} \subset \mathscr{G}^*$ [i.e., the dual space of the stationary subalgebra $\mathscr{G}_{\mathrm{stat}}$ (22)]. Using the formalism of section 2 it can be shown that g_+ and g_- satisfy PBs of the type (23) and (29), respectively. Furthermore (U_0, k_0) constitute a pair of canonically conjugated dynamical variables.

In the case of KM and finite-dimensional groups [2,7] the group element $h \equiv g_- k_0 g_+$ can be shown to obey the required Lie–Poisson structure (61), i.e., the classical prerequisite of a quantum group. It would be interesting to establish this property for general infinite-dimensional groups. Note in this context that (62) provides an extension of notion of "chiral" splitting in $D=2$ conformal models to higher dimensional models with infinite-dimensional Noether symmetry groups.

We would like to emphasize that the method of hamiltonian reduction of T*G (section 2) allowed us to derive in a systematic and general way the fundamental PBs (23) for geometric actions $W[g]$ (30) on coadjoint orbits of arbitrary infinite-dimensional groups.

There are several interesting related problems which can naturally be approached by our method and will be dealt with elsewhere. One of these problems is the understanding in a model-independent way of the role of the "hidden" G_{stat} (21) symmetry in classical exchange algebras.

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